

Reverse Isoperimetric Inequalities In The Plane

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1 Introduction

The *classical isoperimetric inequality* in the Euclidean plane \mathbb{R}^2 states that for a simple closed curve γ of length L , enclosing a region of area A , one gets

$$L^2 - 4\pi A \geq 0, \tag{1.1}$$

and the equality holds if and only if γ is a circle. This fact was known to the ancient Greeks, the first complete mathematical proof was only given in 1882 by Edler[4] (based on the arguments of Steiner[15]). There are various proofs, sharpened forms and generalizations of this inequality.

In 1995, Howard & Treibergs [7] gives a reverse isoperimetric inequality for the plane curves under some assumption on curvature. In Pan & Zhang [13], we have

gotten a reverse isoperimetric inequality

$$L^2 \leq 4\pi(A + |\tilde{A}|), \quad (1.2)$$

where \tilde{A} denotes the oriented area of the domain enclosed by the locus of curvature centers of γ , and the equality holds if and only if γ is a circle.

In this presentation, we state a new reverse isoperimetric inequality for convex curves, which states that if γ is a closed strictly convex curve in the plane \mathbb{R}^2 with length L and enclosing an area A , then we get

$$L^2 \leq 4\pi A + 2\pi|\tilde{A}|, \quad (1.3)$$

where \tilde{A} denotes the oriented area of the domain enclosed by the locus of curvature centers of γ , and the equality holds if and only if γ is a circle.

2 Minkowski's Support function for Convex Plane Curves

From now on, without loss of generality, suppose that γ is a smooth regular positively oriented and closed strictly convex curve in the plane. Take a point O inside γ as the origin of our frame. Let p be the oriented perpendicular distance from O to the tangent line at a point on γ , and θ the oriented angle from the positive x_1 -axis to this perpendicular ray. Clearly, p is a single-valued periodic function of θ with period 2π and γ can be parameterized in terms of θ and $p(\theta)$ as follows

$$\begin{aligned}\gamma(\theta) &= (\gamma_1(\theta), \gamma_2(\theta)) \\ &= (p(\theta) \cos \theta - p'(\theta) \sin \theta, p(\theta) \sin \theta + p'(\theta) \cos \theta),\end{aligned}\tag{2.1}$$

(see for instance [8]). The couple $(\theta, p(\theta))$ is usually called the *polar tangential coordinate* on γ , and $p(\theta)$ its *Minkowski's support function*.

Then, the curvature k of γ can be calculated according to $k(\theta) = \frac{d\theta}{ds} = \frac{1}{p(\theta)+p''(\theta)} > 0$, or equivalently, the radius of curvature ρ of γ is given by

$$\rho(\theta) = \frac{ds}{d\theta} = p(\theta) + p''(\theta). \quad (2.2)$$

Denote L and A the length of γ and the area it bounds, respectively. Then one can get

$$\begin{aligned} L &= \int_{\gamma} ds = \int_0^{2\pi} \rho(\theta) d\theta \\ &= \int_0^{2\pi} p(\theta) d\theta, \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} A &= \frac{1}{2} \int_{\gamma} p(\theta) ds \\ &= \frac{1}{2} \int_0^{2\pi} \left[p^2(\theta) - p'^2(\theta) \right] d\theta. \end{aligned} \quad (2.4)$$

(2.3) and (2.4) are known as *Cauchy's formula* and *Blaschke's formula*, respectively.

3 Some Properties of the Locus of Curvature Centers

We now turn to studying the properties of the locus of curvature centers of a closed strictly convex plane curve γ which is given by (2.1). Let β represent the locus of

centers of curvature of γ . Then $\beta(\theta) = (\beta_1(\theta), \beta_2(\theta))$ can be given by

$$\begin{aligned}\beta(\theta) &= \gamma(\theta) - \rho(\theta)\mathbf{N}(\theta) \\ &= \begin{pmatrix} -p'(\theta)\sin\theta - p''(\theta)\cos\theta, \\ p'(\theta)\cos\theta - p''(\theta)\sin\theta \end{pmatrix},\end{aligned}\tag{3.1}$$

where $\mathbf{N}(\theta) = (\cos\theta, \sin\theta)$ is the unit outward normal vector field along γ .

Proposition 3.1. *The oriented area of the domain enclosed by β is nonpositive. And moreover, if β is simple, then the orientation of β is the reverse direction against that of the original curve γ and the total curvature of β is equal to -2π .*

Proof. To get the claimed results, we calculate the oriented area, denoted by \tilde{A} , of β by Green's formula.

From (3.1), we get

$$\beta_1 d\beta_2 - \beta_2 d\beta_1 = p'(\theta)(p'(\theta) + p'''(\theta))d\theta,$$

and thus \tilde{A} is given by

$$\begin{aligned}\tilde{A} &= \frac{1}{2} \int_{\gamma} \beta_1 d\beta_2 - \beta_2 d\beta_1 = \frac{1}{2} \int_0^{2\pi} p'(\theta)(p'(\theta) + p'''(\theta))d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (p'^2(\theta) - p''^2(\theta))d\theta.\end{aligned}\tag{3.2}$$

Using the Wirtinger inequality for 2π -periodic C^2 real functions gives us $\tilde{A} \leq 0$. If β is simple, then, from Green's formula and the fact that $\tilde{A} \leq 0$, it follows that the orientation of β is the reverse direction against that of γ and the total curvature of β is equal to -2π . \square

The following result is essential to the proof of the main result of this note.

Proposition 3.2. *Let γ be a C^2 closed and strictly convex curve in the plane, ρ the radius of curvature of γ , A the area enclosed by γ and \tilde{A} the oriented area enclosed by β . Then we have*

$$\int_0^{2\pi} \rho^2 d\theta = 2(A + |\tilde{A}|). \quad (3.3)$$

Proof. From (2.2), we have $p'' = \rho - p$, and thus,

$$p''^2 = \rho^2 - 2p\rho + p^2 = \rho^2 - 2p(p + p'') + p^2 = \rho^2 - 2pp'' - p^2.$$

Now, according to (3.2), $|\tilde{A}|$ can be rewritten as

$$\begin{aligned}
|\tilde{A}| &= \frac{1}{2} \int_0^{2\pi} (\rho^2 - 2pp'' - p^2 - p'^2) d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \rho^2 - \int_0^{2\pi} pp'' d\theta - \frac{1}{2} \int_0^{2\pi} (p^2 + p'^2) d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \rho^2 d\theta - pp'|_0^{2\pi} + \int_0^{2\pi} p'^2 d\theta \\
&\quad - \frac{1}{2} \int_0^{2\pi} (p^2 + p'^2) d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \rho^2 d\theta + \frac{1}{2} \int_0^{2\pi} (p'^2 - p^2) d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \rho^2 d\theta - A,
\end{aligned}$$

which completes the proof. \square

We remark that the equality (3.3) is new, and it would be interesting to find a similar formula for higher dimensional convex surfaces.

4 The Unit-speed Outward Normal Flow

Let $\gamma(\theta, t)$ be a family of closed convex plane curves with initial curve $\gamma_0(\theta)$. In this case, the first author of the present paper has shown in [12] that the tangent vector field \mathbf{T} and the unit outward normal vector field \mathbf{N} are independent of the time t . And thus the evolution

problem in question can be expressed as follows

$$\begin{cases} \frac{\partial \gamma(\theta, t)}{\partial t} = \mathbf{N}(\theta) \\ \gamma(\theta, 0) = \gamma_0(\theta). \end{cases} \quad (4.1)$$

Lemma 4.1. Under the evolution defined by (4.1), let $\gamma(\theta, t)$ be the curve at time $t \geq 0$, we have the following formulas:

$$\rho(\theta, t) = \rho(\theta, 0) + t; \quad (4.2)$$

$$k(\theta, t) = \frac{k(\theta, 0)}{1 + k(\theta, 0)t}; \quad (4.3)$$

$$L(t) = L(0) + 2\pi t; \quad (4.4)$$

$$A(t) = A(0) + L(0)t + \pi t^2, \quad (4.5)$$

where $\rho(\theta, t)$ and $k(\theta, t)$ are the radius of curvature and the curvature, $L(t)$ and $A(t)$ are the length of the evol-

ing curve and the area it encloses at time t , respectively.
 \square

(4.5) is usually called *the Steiner polynomial* for the evolving curve. It is easy to check that the isoperimetric defect $L^2 - 2\pi A$ of the evolving curve is invariant under the unit-speed outward normal follow.

5 A New Isoperimetric Inequality

Theorem 3.1 For a C^2 closed and strictly convex curve γ , L and A are the length of γ and the area it encloses, one gets

$$\int_0^{2\pi} \rho^2(\theta) d\theta \geq \frac{L^2 - 2\pi A}{\pi} \quad (5.1)$$

And moreover, the equality in (5.1) holds if and only if γ is a circle.

Proof. It is obvious that the equality in (5.1) holds when γ is a circle. If we can prove that

$$\int_0^{2\pi} \rho^2(\theta) d\theta > \frac{L^2 - 2\pi A}{\pi}$$

when γ is not a circle, then the result holds. This can be concluded by proving the following theorem.

Theorem 5.2 If γ is a C^2 closed strictly convex and non-circular curve in the plane, then

$$\int_0^{2\pi} \rho^2(\theta) d\theta > \frac{L^2 - 2\pi A}{\pi} \tag{5.2}$$

holds.

To prove Theorem 5.2, we need some definitions.

Definition 5.3 Let $t_1 \geq t_2$ be the roots of the Steiner polynomial $A(t)$, r_i and r_e be the radii of the largest inscribed and the smallest circumscribed circles of γ (called the **inradius** and the **outradius** of γ), respectively. Let k be the curvature of γ , $\rho = \frac{1}{k}$ the radius of curvature, and ρ_{\max} and ρ_{\min} the maximum and the minimum values of ρ . These quantities are all equal if the curve γ is a circle.

Lemma 5.4 If γ is convex and non-circle, then

$$-\rho_{\max} < t_2 < -r_e < -\frac{L}{2\pi} < -r_i < t_1 < -\rho_{\min}. \quad (5.3)$$

Proof. See Green and Osher [6].

Definition 5.5 Consider

$$\sup\{\int_I \rho(\theta)d\theta | I \subset S^1, \int_I d\theta = \pi\}.$$

Let I_1 denote a subset of S^1 of measure π realizing this bound, and let I_2 be its complement. There exists a real number a such that

$$I_1 \subseteq \{\theta | \rho(\theta) \geq a\}, \quad I_2 \subseteq \{\theta | \rho(\theta) \leq a\}.$$

We set

$$\rho_1 = \frac{1}{\pi} \int_{I_1} \rho(\theta)d\theta, \quad \rho_2 = \frac{1}{\pi} \int_{I_2} \rho(\theta)d\theta,$$

Note that

$$\rho_1 + \rho_2 = \frac{L}{\pi}, \quad \rho_1 \geq \rho_2.$$

Proposition 5.6 Let γ be a strictly convex curve, if it is not a circle, then

$$\rho_1 > \rho_2.$$

In other words, there exists a real number $b > 0$ such that

$$\rho_1 = \frac{L}{2\pi} + b, \quad \rho_2 = \frac{L}{2\pi} - b.$$

Proposition 5.7 For γ a symmetric strictly convex curve and not a circle, then

$$\rho_1 > -t_2.$$

Proposition 5.8 For γ a strictly convex curve and not a circle, then

$$\rho_1 > -t_2.$$

The following two elementary lemmata have appeared in Green and Osher [6], we omitted their proofs here.

Lemma 5.9 Let $F(x)$ be a convex function on $(0, +\infty)$, then

$$\frac{1}{2\pi} \int_{S^1} F(\rho(\theta)) d\theta \geq \frac{1}{2} [F(\rho_1) + F(\rho_2)].$$

Lemma 5.10 If $F(x)$ is strictly convex on $(0, +\infty)$, then for $b > a > 0$ and c arbitrary, one gets

$$F(c - a) + F(c + a) < F(c - b) + F(c + b).$$

□

Proof of Theorem 5.2. We already have that

$$\frac{1}{2\pi} \int_{S^1} F(\rho(\theta)) d\theta \geq \frac{1}{2} [F(\rho_1) + F(\rho_2)],$$

Now

$$\begin{aligned}\rho_1 &= \frac{L}{2\pi} + b, & \rho_2 &= \frac{L}{2\pi} - b, \\ -t_1 &= \frac{L}{2\pi} - u, & -t_2 &= \frac{L}{2\pi} + u.\end{aligned}$$

By Proposition 5.8, $b > u > 0$, and so by Lemma 5.10,

$$F(\rho_1) + F(\rho_2) > F(-t_1) + F(-t_2). \quad (5.4)$$

Takeing $F(x) = x^2$ and using Lemma 5.9 we get

$$\frac{1}{2\pi} \int_0^{2\pi} \rho^2(\theta) d\theta \geq \frac{1}{2}(\rho_1^2 + \rho_2^2).$$

The above inequality (5.4) is

$$\rho_1^2 + \rho_2^2 > t_1^2 + t_2^2,$$

and t_1, t_2 are the roots of $A(t) = \pi t^2 + Lt + A = 0$. Thus

$$t_1^2 + t_2^2 = \frac{L^2 - 2\pi A}{\pi^2}.$$

All the results indicate that

$$\int_0^{2\pi} \rho^2(\theta) d\theta > \frac{L^2 - 2\pi A}{\pi}.$$

This proves the theorem. □

Theorem 5.11. (A New Reverse Isoperimetric Inequality) *If γ is a closed strictly convex plane curve with length L and enclosing an area A , let \tilde{A} denote the oriented area bounded by its locus of centers of curvature, then we get*

$$L^2 \leq 4\pi A + 2\pi|\tilde{A}|, \tag{5.5}$$

where the equality holds if and only if γ is a circle. \square

The following corollary is a direct consequence of the classical isoperimetric inequality (1.1) and our reverse isoperimetric inequality (5.5).

Corollary 5.12. *Let β be the locus of curvature centers of a closed strictly convex plane curve γ . Then the oriented area \tilde{A} of β is zero if and only if γ is a circle and thus β is a point which is the center of γ . \square*

6 Final Remarks

It should be pointed out that the above reverse isoperimetric inequalities (1.2) and (1.3) are obtained by the integration of the radius of curvature, our curves must

be strictly convex. We wonder if this sort of inequalities can be obtained for any (simple) closed plane curves. And furthermore, it would be interesting to generalize these inequalities to higher dimensional spaces.

Another problem is that if there is a best constant C such that

$$L^2 \leq 4\pi A + C|\tilde{A}|, \quad (6.1)$$

where the equality holds if and only if γ is a circle.

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